

On Some t -($2k, k, \lambda$) Designs

RYUZABURO NODA

*Department of Mathematics, College of General Education,
Osaka University, Toyonaka, Osaka, Japan*

Communicated by Marshall Hall, Jr.

Received September 29, 1976

t -($2k, k, \lambda$) designs having a property similar to that of Hadamard 3-designs are studied. We consider conditions (i), (ii), or (iii) for t -($2k, k, \lambda$) designs: (i) The complement of each block is a block. (ii) If A and B are a complementary pair of blocks, then $|A \cap C| = |B \cap C| \pm u$ holds for any block C distinct from A and B , where u is a positive integer. (iii) if A and B are a complementary pair of blocks, then $|A \cap C| = |B \cap C|$ or $|A \cap C| = |B \cap C| \pm u$ holds for any block C distinct from A and B , where u is a positive integer. We show that a t -($2k, k, \lambda$) design with $t \geq 2$ and with properties (i) and (ii) is a 3 -($2u(2u+1)$, $u(2u+1)$, $u(2u^2+u-2)$) design, and that a t -($2k, k, \lambda$) design with $t \geq 4$ and with properties (i) and (iii) is the 5 -($12, 6, 1$) design, the 4 -($8, 4, 1$) design, a 5 -($2u^2, u^2, \frac{1}{4}(u^2-3)(u^2-4)$) design, or a 5 -($\frac{3}{8}u(2u+1)$, $\frac{1}{4}u(2u+1)$, $\frac{1}{8}u(2u^2+u-9)(2u^2+u-12)$) design.

1. INTRODUCTION

We understand a t -(v, k, λ) design to be a pair of a v -set X and a family \mathcal{B} of k -subsets of X , such that each t -subset of X is contained in λ elements of \mathcal{B} . Elements of X and \mathcal{B} are called points and blocks, respectively. In this paper we consider t -(v, k, λ) designs with $v = 2k$ and with the property that

- (i) the complement of each block is a block.

Alltop (Theorem A₁ [1]) showed that if t is even and $k > t$ then a t -($2k, k, \lambda$) design with property (i) is already a $(t+1)$ -design. Using this result we can easily prove the following.

PROPOSITION 0. *Let (X, \mathcal{B}) be a t -($2k, k, \lambda$) design with $t \geq 2$. Assume that (X, \mathcal{B}) satisfies (i) and the following;*

(*) *if A and B are a complementary pair of blocks, then $|A \cap C| = |B \cap C|$ holds for any block C distinct from A and B .*

Then (X, \mathcal{B}) is a 3 -($2k, k, \frac{1}{2}(k-2)$) design, namely, a Hadamard 3-design. In particular $t \leq 3$.

In this paper we consider condition (ii) or (iii) for t -($2k, k, \lambda$) designs in place of (*).

(ii) If A and B are a complementary pair of blocks, then $|A \cap C| = |B \cap C| \pm u$ holds for any block C distinct from A and B . Here u is a positive integer, which may be dependent of the choice of A and B .

(iii) If A and B are a complementary pair of blocks, then $|A \cap C| = |B \cap C|$ or $|A \cap C| = |B \cap C| \pm u$ holds for any block C distinct from A and B , where u is as in (ii).

Our results are as follows.

THEOREM 1. *Let (X, \mathcal{B}) be a t -($2k, k, \lambda$) design with $t \geq 2$. Assume that (X, \mathcal{B}) satisfies conditions (i) and (ii). Then $t \leq 3$, u in (ii) is independent of the choice of A, B and (X, \mathcal{B}) is a 3 -($2u(2u+1), u(2u+1), u(2u^2+u-2)$) design.*

THEOREM 2. *Let (X, \mathcal{B}) be a t -($2k, k, \lambda$) design with $t \geq 4$. Assume that (X, \mathcal{B}) satisfies conditions (i) and (iii). Then $t \leq 5$, u in (iii) is independent of the choice of A, B , u is even, and one of the following holds;*

- (1) (X, \mathcal{B}) is the 5-(12, 6, 1) design,
- (2) (X, \mathcal{B}) is a 5-($\frac{2}{3}u(2u+1), \frac{1}{3}u(2u+1), \frac{1}{54}u(2u^2+u-9)(2u^2+u-12)$) design,
- (3) (X, \mathcal{B}) is a 4-($2u^2, u^2, \frac{1}{2}(u^2-2)(u^2-3)$) design (5-($2u^2, u^2, \frac{1}{4}(u^2-3)(u^2-4)$) design if $u > 2$).

The only known example of 3-designs in Theorem 1 is the trivial 3-(6, 3, 1) design. The 5-(12, 6, 1) design exists and is unique [3]. The only known example of 5-designs in Theorem 2(2) is the 5-(24, 12, 48) design associated with the Mathieu group M_{24} (The author does not know whether this is a unique 5-(24, 12, 48) design.) The only known example of 4-designs in Theorem 2(3) is the trivial 4-(8, 4, 1) design.

Our notation and terminology for t -designs are standard. The number of blocks of a t -(v, k, λ) design containing fixed i points is denoted by λ_i ($0 \leq i \leq t$), $\lambda_0 = b$ being the total number of blocks. As is well known equalities $\lambda_i = ((v-i)/(k-i))\lambda_{i+1}$ ($0 \leq i < t$) hold. If (X, \mathcal{B}) is a t -(v, k, λ) design and x is a point of X then $(X - \{x\}, \{B - \{x\} \mid x \in B \in \mathcal{B}\})$ is a $(t-1)$ -($v-1, k-1, \lambda$) design. This design is called a contraction of (X, \mathcal{B}) .

2. PROOFS OF THEOREMS 1 AND 2

We begin with the following general result.

LEMMA 3. *Let a t -(v, k, λ) design (X, \mathcal{B}) have a disjoint pair of blocks A and B . For $C \in \mathcal{B} - \{A, B\}$ set $\alpha_C = |A \cap C|$ and $\beta_C = |B \cap C|$. Then*

(1) if $t \geq 2$, $\sum_C (\alpha_C - \beta_C)^2 = 2k(\lambda_1 - \lambda_2 - k)$, where C runs over all blocks distinct from A and B , and

(2) if $t \geq 4$, $\sum_C (\alpha_C - \beta_C)^4 = 2k\{(\lambda_1 - \lambda_2) + 6(k-1)(\lambda_2 - 2\lambda_3 + \lambda_4) - k^3\}$.

Proof of (1). Let C_i ($1 \leq i \leq b-2$) denote blocks distinct from A and B , and for brevity set $\alpha_i = \alpha_{C_i}$ and $\beta_i = \beta_{C_i}$. Counting in two ways the number of pairs (a, C) with $a \in A \cap C$ (or $B \cap C$) gives

$$\sum_{i=1}^{b-2} \alpha_i = \sum_{i=1}^{b-2} \beta_i = k(\lambda_1 - 1), \quad (2.1)$$

and counting the number of triples (a, a', C) with $a, a' \in A \cap C$ (or $B \cap C$), $a \neq a'$ gives

$$\sum_{i=1}^{b-2} \alpha_i(\alpha_i - 1) = \sum_{i=1}^{b-2} \beta_i(\beta_i - 1) = k(k-1)(\lambda_2 - 1). \quad (2.2)$$

Also counting in two ways the number of triples (a, e, C) with $a \in A \cap C$ and $e \in B \cap C$ gives

$$\sum_{i=1}^{b-2} \alpha_i \beta_i = k^2 \lambda_2. \quad (2.3)$$

By (2.1) ~ (2.3) we obtain

$$\sum_{i=1}^{b-2} (\alpha_i - \beta_i)^2 = 2k(\lambda_1 - \lambda_2 - k).$$

Proof of (2). Counting arguments similar to those above yield

$$\begin{aligned} \sum_{i=1}^{b-2} \alpha_i(\alpha_i - 1)(\alpha_i - 2) &= \sum_{i=1}^{b-2} \beta_i(\beta_i - 1)(\beta_i - 2) \\ &= k(k-1)(k-2)(\lambda_3 - 1), \end{aligned} \quad (2.4)$$

$$\begin{aligned} \sum_{i=1}^{b-2} \alpha_i(\alpha_i - 1)(\alpha_i - 2)(\alpha_i - 3) &= \sum_{i=1}^{b-2} \beta_i(\beta_i - 1)(\beta_i - 2)(\beta_i - 3) \\ &= k(k-1)(k-2)(k-3)(\lambda_4 - 1), \end{aligned} \quad (2.5)$$

$$\sum_{i=1}^{b-2} \alpha_i(\alpha_i - 1) \beta_i = \sum_{i=1}^{b-2} \beta_i(\beta_i - 1) \alpha_i = k^2(k-1) \lambda_3, \quad (2.6)$$

$$\begin{aligned} \sum_i \alpha_i(\alpha_i - 1)(\alpha_i - 2) \beta_i &= \sum_i \beta_i(\beta_i - 1)(\beta_i - 2) \alpha_i \\ &= k^2(k-1)(k-2) \lambda_4, \end{aligned} \quad (2.7)$$

$$\sum \alpha_i(\alpha_i - 1) \beta_i(\beta_i - 1) = k^2(k-1)^2 \lambda_4. \quad (2.8)$$

By (2.1)–(2.8) we obtain

$$\begin{aligned}
 \sum_{i=1}^{b-2} \alpha_i^2 &= \sum_i \beta_i^2 = k\{\lambda_1 + (k-1)\lambda_2 - k\}, \\
 \sum_i \alpha_i^3 &= \sum_i \beta_i^3 = k\{\lambda_1 + 3(k-1)\lambda_2 + (k-1)(k-2)\lambda_3 - k^2\}, \\
 \sum_i \alpha_i^2 \beta_i &= \sum_i \alpha_i \beta_i^2 = k^2\{\lambda_2 + (k-1)\lambda_3\}, \\
 \sum_i \alpha_i^4 &= \sum_i \beta_i^4 = k\{\lambda_1 + 7(k-1)\lambda_2 + 6(k-1)(k-2)\lambda_3 \\
 &\quad + (k-1)(k-2)(k-3)\lambda_4 - k^3\}, \\
 \sum_i \alpha_i^3 \beta_i &= \sum_i \alpha_i \beta_i^3 = k^2\{\lambda_2 + 3(k-1)\lambda_3 + (k-1)(k-2)\lambda_4\}, \\
 \sum_i \alpha_i^2 \beta_i^2 &= k^2\{\lambda_2 + 2(k-1)\lambda_3 + (k-1)^2\lambda_4\},
 \end{aligned}$$

and consequently

$$\sum_i (\alpha_i - \beta_i)^4 = 2k\{(\lambda_1 - \lambda_2) + 6(k-1)(\lambda_2 - 2\lambda_3 + \lambda_4) - k^3\}.$$

Proof of Theorem 1. (X, \mathcal{B}) is a 3-design by Theorem A₁ [1]. Let A and $B = X - A$ be blocks and α_C, β_C as in Lemma 3. Then by the assumption on (X, \mathcal{B}) and Lemma 3 we have

$$(b-2)u^2 = \sum_C (\alpha_C - \beta_C)^2 = 2k(\lambda_1 - \lambda_2 - k). \quad (2.9)$$

Hence u is independent of the choice of A and B . Now since $\lambda_2 = ((2k-2)/(k-2))\lambda_3$ is an integer, $k-2$ divides $2\lambda_3$. Set

$$e = 2\lambda_3/(k-2). \quad (2.10)$$

Then $\lambda_2 = (k-1)e$, $\lambda_1 = (2k-1)e$, and $b = 2(2k-1)e$. Putting these in (2.9) gives

$$u^2 = \frac{2k(\lambda_1 - \lambda_2 - k)}{b-2} = \frac{k^2(e-1)}{2ke - (e+1)}. \quad (2.11)$$

Then since $(2ke - (e+1), k(e-1)) = (2ke - (e+1) - 2k(e-1), k(e-1)) = (2k - (e+1), k(e-1))$ and $(2ke - (e+1), k) = (2ke - (e+1) - 2ke, k) = (e+1, k)$, it follows from (2.11) that $2ke - (e+1)$ divides $(e+1)(2k - e - 1) = 2ke - (e+1) + 2k - e(e+1)$. We claim that $2k - e(e+1) = 0$. Suppose first that $2k > e(e+1)$. Then since $2ke - (e+1)$ divides $2k - e(e+1)$ it follows that $2ke - (e+1) \leq 2k - e(e+1)$, hence $(e-1)(2k + e + 1) \leq 0$, hence $e = 1$, hence $u^2 = 0$ by (2.11), a contradiction. Now if $2k < e(e+1)$ it follows that $2ke - (e+1) \leq e(e+1) - 2k$, hence $(e+1)(2k - e - 1) \leq 0$, hence

$$2k \leq e + 1. \quad (2.12)$$

On the other hand applying a result of Cameron [2, Theorem 1] to a contraction (a 2 -($2k-1, k-1, \lambda_3$) design) of (X, \mathcal{B}) we obtain $\lambda_1 \leq \binom{2k-1}{2}$, hence $e \leq k-1$. This contradicts (2.12). Hence it follows that $2k = e(e+1)$, so $u^2 = \frac{1}{4}e^2$ by (2.11), and $\lambda_3 = u(2u^2 + u - 2)$ by (2.10). Finally since $((k-3)/(2k-3))\lambda_3 = (2u^2 + u - 3)u(2u^2 + u - 2)/(4u^2 + 2u - 3)$ is not an integer we have $t \leq 3$.

Proof of Theorem 2. (X, \mathcal{B}) is a 5-design by Theorem A₁ [1]. We make use of arguments similar to those in the proof of Theorem 1. Let A, B and α_C, β_C be as in the above. Then since $t \geq 4$ it follows by Theorem 1 that $\alpha_C = \beta_C$ for some block C . Hence k and u are even. Let d be the number of blocks C such that $|\alpha_C - \beta_C| = u$. Then by Lemma 3 we have

$$2k(\lambda_1 - \lambda_2 - k) = \sum_C (\alpha_C - \beta_C)^2 = u^2 d$$

and

$$2k\{(\lambda_1 - \lambda_2) + 6(k-1)(\lambda_2 - 2\lambda_3 + \lambda_4) - k^3\} = \sum_C (\alpha_C - \beta_C)^4 = u^4 d.$$

Then since $d \neq 0$ by Proposition O it follows that

$$u^2 = \frac{(\lambda_1 - \lambda_2) + 6(k-1)(\lambda_2 - 2\lambda_3 + \lambda_4) - k^3}{\lambda_1 - \lambda_2 - k}. \quad (2.13)$$

Now since $\lambda_3 = ((2k-3)/(k-3))\lambda_4$ and $\lambda_2 = ((2k-2)(2k-3)/(k-2)(k-3))\lambda_4$ are integers we have that $k-3$ divides $3\lambda_4$, $k-2$ divides $2\lambda_4$, whence $(k-2)(k-3)$ divides $6\lambda_4$. Set

$$e = \frac{6\lambda_4}{(k-2)(k-3)} = \frac{12\lambda_5}{(k-3)(k-4)}. \quad (2.14)$$

Then $\lambda_3 = ((2k-3)/(k-3))\lambda_4 = \frac{1}{6}(2k-3)(k-2)e$, $\lambda_2 = ((2k-2)/(k-2))\lambda_3 = \frac{1}{3}(k-1)(2k-3)e$, and $\lambda_1 = ((2k-1)/(k-1))\lambda_2 = \frac{1}{3}(2k-1)(2k-3)e$. Putting these in (2.13) gives

$$\begin{aligned} u^2 &= \frac{\frac{1}{3}k(2k-3)e + 6(k-1)e(\frac{1}{6}k^2 - \frac{1}{6}k) - k^3}{\frac{1}{3}k(2k-3)e - k} \\ &= \frac{(2k-3)e + 3(k-1)^2e - 3k^2}{(2k-3)e - 3} \\ &= \frac{k\{(3k-4)e - 3k\}}{(2k-3)e - 3} = \frac{k\{3k(e-1) - 4e\}}{2ke - 3(e+1)}. \end{aligned} \quad (2.15)$$

Then since $(2ke - 3(e+1), k) = (3(e+1), k)$ and $(2ke - 3(e+1), 2\{3k(e-1) - 4e\}) = (2ke - 3(e+1), e+9-6k)$ it follows by (2.15)

that $2ke - 3(e + 1)$ divides $3(e + 1)(6k - e - 9) = 3\{(6ke - 9e - 9) + (6k - e^2 - e)\}$. We treat the following three cases separately.

Case 1. $6k < e^2 + e$. We have that $2ek - 3(e + 1)$ divides $3(e^2 + e - 6k)$. Suppose first that $3(e^2 + e - 6k) = 2ek - 3(e + 1)$. Then $k = 3(e + 1)^2/2(e + 9)$, so $u^2 = (e + 1)(9e + 1)/4(e + 9)$ by (2.15). Hence $2(e + 9)$ divides $(3(e + 1)^2, (e + 1)(9e + 1)) = (e + 1)(3(e + 1), 9e + 1) = (e + 1)(3(e + 1), 8)$, so $(e + 9)$ divides $4(e + 1)$. Then $e = 7$ or 23 and hence $u^2 = 8$ or 39 , a contradiction. Consequently we have that $3(e^2 + e - 6k) \geq 2\{2ek - 3(e + 1)\}$, so

$$3(e + 1)(e + 2) \geq k(4e + 18) \quad (2.16)$$

On the other hand applying the result of Cameron [2, Theorem 1] to a contraction (a $4(2k - 1, k - 1, \lambda_5)$ design) of (X, \mathcal{B}) we obtain $\lambda_1 \leq \binom{2k-1}{3}$, hence $e + 1 \leq k$. Putting this in (2.16) gives $3(e + 2) \geq 4e + 18$, a contradiction.

Case 2. $6k > e^2 + e$. We have $3(6k - e^2 - e) \geq 2ek - 3(e + 1)$, hence $-3(e^2 - 1) \geq 2k(e - 9)$, hence $e \leq 8$. We check each value of e . Let first $e = 1$. Then by (2.15) $u^2 = -2k/(k - 3) < 0$, a contradiction. If $e = 2$, then $u^2 = k(3k - 8)/(4k - 9)$. Hence $4k - 9$ divides $k + 9$. Hence $k = 6$, and $\lambda_5 = 1$ by (2.14). If $e = 3$, then $u^2 = k$ and $\lambda_4 = \frac{1}{2}(k - 2)(k - 3)$. This gives a series of parameters in Theorem 2(3). Now let $e = 4$. Then $u^2 = k(9k - 16)/(8k - 15)$. Hence $8k - 15$ divides $k - 15$, a contradiction. The same argument rules out the cases $e = 5, 6, 7$ and gives $k = 12$ in the case $e = 8$. But since $6 \cdot 12 = 8^2 + 8$, $(k, e) = (12, 8)$ should be treated in the following case.

Case 3. $6k = e^2 + e$. In this case it follows from (2.15) that $u^2 = \frac{1}{4}e^2$. Hence $k = \frac{1}{6}e(e + 1) = \frac{1}{3}u(2u + 1)$, hence $\lambda_5 = \frac{1}{24}u(2u^2 + u - 9)(2u^2 + u - 12)$ by (2.14).

Finally since $((k - 5)/(2k - 5))\lambda_5$ is not an integer for any parameters in Theorem 2 it follows that $t \leq 5$.

Note. By a result of Cameron [2, Theorem 1] the dual of a contraction \mathcal{D}' of (X, \mathcal{B}) in Theorem 2 is a partial design with class number 3. Let N_1, N_2 , and N_3 be the matrices whose rows and columns are indexed by the blocks of \mathcal{D}' and whose (C, C') entry is 1 if $|C \cap C'| = \frac{1}{2}(k - u) - 1, \frac{1}{2}k - 1$ and $\frac{1}{2}(k + u) - 1$, respectively, and 0 otherwise. Using Cameron's method we can describe eigenvalues of N_i and their multiplicities in terms of design parameters and block intersection numbers, and can show that N_2 has $(k^2/u^2) - 1$ as its eigenvalue. Then since $(k^2/u^2) - 1$ must be an integer it follows that u divides k . Therefore a 5-design with the parameters in Theorem 2(2) exists only if $u \equiv 4 \pmod{6}$.

The author tried to obtain other necessary conditions from the matrices N_i for a 5-design in Theorem 2(2), (3) to exist, but did not succeed. For example, trace $N_i = 0$ ($1 \leq i \leq 3$) are trivially satisfied and give no restrictions on k and u .

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